

Laplace's and Poisson's Equations, Uniqueness Theorem and Method of Images

In electrostatics, the electric field $\mathbf{E}$ and electrostatic potential V are related by $\mathbf{E} = -\nabla V$

The electric field is also related to the charge density ρ through **Gauss's law**, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$

where ϵ_0 is the permittivity of free space.

Substituting $\mathbf{E} = -\nabla V$, $\nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0}$

Therefore, $\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$

This is called **Poisson's equation**.

In Cartesian coordinates,

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon_0}}$$

Poisson's equation determines the electrostatic potential V when the charge distribution ρ is known. Once V is obtained, the electric field can be calculated from $\boxed{\mathbf{E} = -\nabla V}$

Laplace's Equation:

In a charge-free region, $\rho = 0$

Therefore, Poisson's equation becomes $\nabla^2 V = 0$

This is called **Laplace's equation**.

Laplace's equation applies to any region where there is **no volume charge**.

The solution is determined by the **boundary conditions**.

Laplace's Equation in Different Coordinate Systems

In Cartesian coordinates, $\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$

Spherical coordinates: For a general potential $V(r, \theta, \phi)$,

$$\boxed{\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0}$$

For spherical symmetry, $V = V(r)$, so

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right) = 0 \quad \text{or} \quad \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right) = 0$$

Integrating, $r^2 \frac{dV}{dr} = A$

Therefore, $\frac{dV}{dr} = \frac{A}{r^2}$ and $\boxed{V = -\frac{A}{r} + B}$

Boundary Conditions for Conductors

In electrostatic equilibrium:

1. The electric field inside a conductor is zero: $\boxed{\mathbf{E} = 0}$
2. The potential throughout a conductor is constant.
3. The electric field immediately outside a conductor is normal to its surface.
4. For an earthed conductor, $\boxed{V = 0}$ at the conducting surface.

These boundary conditions are extremely important in applying Laplace's equation and the method of images.

Uniqueness Theorem

The uniqueness theorem states that:

If the potential V is specified on the boundaries of a charge-free region, then there exists only one solution of Laplace's equation satisfying those boundary conditions.

More generally, if the charge density ρ throughout a region and the potential on its boundary are specified, the solution of Poisson's equation is unique.

This theorem provides the mathematical justification for the **method of images**.

We may replace the original charge-conductor arrangement by an imaginary arrangement of charges, provided that:

The potential satisfies Poisson's/Laplace's equation in the region of interest.

And the potential satisfies the same boundary conditions.

According to the uniqueness theorem, the potential obtained must then be the **actual physical potential** in that region.

Method of Images

It is a technique used to solve electrostatic problems involving conductors.

Instead of directly calculating the charge induced on a conductor, we replace the conductor by suitable **imaginary charges**, called **image charges**.

The image charges are chosen so that the conducting surface satisfies the required boundary condition.

For an earthed conductor, $V = 0$ on its surface.

The potential produced by the real charge together with the image charge therefore gives the correct potential in the region outside the conductor.

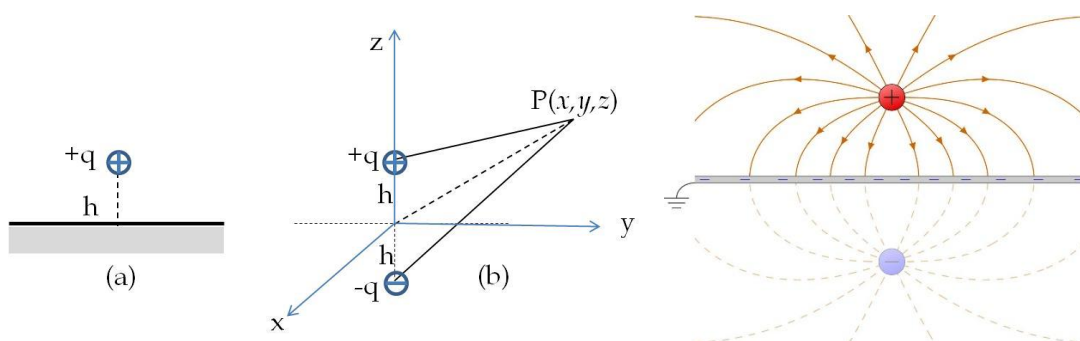
Image charges are **mathematical charges**. They do not actually exist inside the conductor.

Point Charge in Front of an Earthed Infinite Conducting Plane

Consider an infinite conducting plane located at $x = 0$ & earthed so that $V = 0$ on the plane.

Let a point charge q be placed at $x = a$ in front of the conducting plane.

We replace the conducting plane by an image charge $\boxed{-q}$ located at $x = -a$.



The arrangement is therefore: $\boxed{q \text{ at } x = a, -q \text{ at } x = -a}$

Potential at any point: Let the observation point be $P(x, y, z)$.

The distance from q to P is $r_1 = \sqrt{(x - a)^2 + y^2 + z^2}$ and

the distance from the image charge $-q$ is $r_2 = \sqrt{(x + a)^2 + y^2 + z^2}$.

Therefore,
$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_1} - \frac{q}{r_2} \right) \quad \text{or} \quad V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right]$$

Verification of boundary condition

On the conducting plane, $x = 0$. Then $r_1 = r_2$ and consequently $V = 0$.

Thus the plane is indeed at zero potential.

By the uniqueness theorem, this potential is the actual potential in the region $x > 0$.

Force on the Point Charge

The force on the real charge q is the same as the Coulomb force between q and its image charge $-q$. The separation between them is $2a$.

Therefore, $F = \frac{1}{4\pi\epsilon_0} \frac{q^2}{(2a)^2}$ Hence,
$$F = \frac{q^2}{16\pi\epsilon_0 a^2}$$

The force is directed towards the conducting plane.
$$F = -\frac{q^2}{16\pi\epsilon_0 a^2} \hat{x}$$
 for a charge at $x = a$.

Surface Charge Density on the Conducting Plane

The surface charge density can be obtained from $\sigma = \epsilon_0 E_n$

where E_n is the normal electric field immediately outside the conductor.

For a point on the plane at radial distance ρ from the point directly below the charge,

$$\sigma(\rho) = -\frac{qa}{2\pi(\rho^2 + a^2)^{3/2}}$$

The negative sign indicates that the induced charge on the grounded plane is opposite in sign to the point charge.

Point Charge in Front of an Earthed Conducting Sphere

Now consider a conducting sphere of radius R , maintained at zero potential.

Let a point charge q be placed outside the sphere at a distance a from its centre, where $a > R$. We use the method of images to replace the sphere by a suitable image charge inside it.

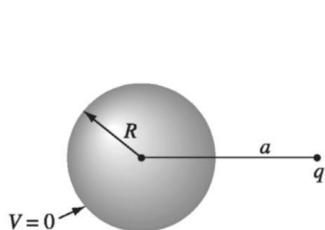


FIGURE 3.12

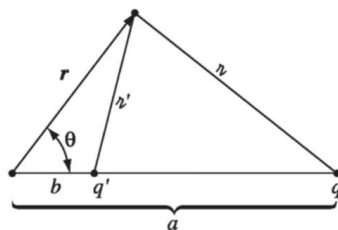
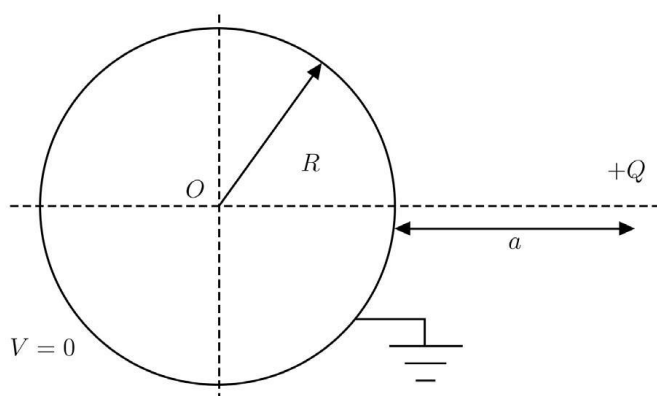
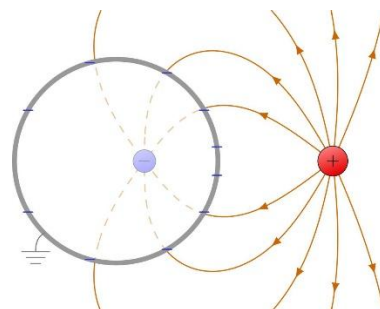


FIGURE 3.13



The image charge is $q' = -\frac{R}{a}q$ and its distance from the centre of the sphere is $b = \frac{R^2}{a}$

Thus, $q' = -\frac{Rq}{a}$, $b = \frac{R^2}{a}$

The image charge lies on the line joining the centre of the sphere to the real charge.

Derivation of the Image Charge

Let the centre of the sphere be O .

The real charge q is at distance a from O , while the image charge q' is at distance b from O .

For any point P on the sphere, $OP = R$.

Let the angle between OP and the line joining O to q be θ .

The distance from P to the real charge is $r_1 = \sqrt{a^2 + R^2 - 2aR \cos \theta}$.

The distance from P to the image charge is $r_2 = \sqrt{b^2 + R^2 - 2bR \cos \theta}$.

Choose $b = \frac{R^2}{a}$. Then $r_2 = \frac{R}{a}r_1$. Therefore, $\frac{q'}{r_2} = \frac{q'}{(R/a)r_1}$.

To make the potential zero on the sphere, we require $\frac{q}{r_1} + \frac{q'}{r_2} = 0$.

Hence, $q' = -q \frac{R}{a}$. Thus, $q' = -\frac{Rq}{a}$ and $b = \frac{R^2}{a}$.

The potential at any point outside the sphere is $V = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_1} + \frac{q'}{r_2} \right)$ where $q' = -\frac{Rq}{a}$, $b = \frac{R^2}{a}$.

Therefore, $V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r_1} - \frac{R/a}{r_2} \right]$ On the sphere, $r_2 = \frac{R}{a}r_1$, so $V = 0$.

Thus the spherical surface is at zero potential. By the uniqueness theorem, this is the actual electrostatic potential outside the grounded sphere.

Force on the Point Charge

The force on the real charge is obtained by considering its interaction with the image charge.

The separation between q and q' is $a - b$, where $b = \frac{R^2}{a}$.

Therefore, $F = \frac{1}{4\pi\epsilon_0} \frac{|qq'|}{(a-b)^2}$. Substituting $q' = -\frac{Rq}{a}$,

we get $F = \frac{1}{4\pi\epsilon_0} \frac{q^2 R/a}{\left(a - \frac{R^2}{a}\right)^2}$. Since $a - \frac{R^2}{a} = \frac{a^2 - R^2}{a}$,

we obtain

$$F = \frac{1}{4\pi\epsilon_0} \frac{q^2 R a}{(a^2 - R^2)^2}$$

The force is directed towards the sphere.

Comparison: Infinite Plane and Sphere

Feature	Earthed Infinite Plane	Earthed Sphere
Real charge	q	q
Image charge	$-q$	$-qR/a$
Position of image	Same distance behind plane	R^2/a from centre
Boundary	$V = 0$	$V = 0$
Image charge physical?	No	No
Force direction	Towards plane	Towards sphere

Short Exam Questions

1. Derive Poisson's equation from Gauss's law.
2. Derive Laplace's equation from Poisson's equation.
3. State and explain the uniqueness theorem.
4. What is meant by the method of images?
5. Explain why an image charge can be used in electrostatic problems.
6. Find the image charge for a point charge near an earthed infinite conducting plane.
7. Calculate the force between a point charge and an earthed conducting plane.
8. Find the image charge and its position for a point charge outside an earthed conducting sphere.
9. Derive the potential outside a grounded conducting sphere.
10. Obtain the force acting on a point charge placed outside an earthed conducting sphere.

Key idea to remember

Poisson's equation tells us how charge density determines potential; Laplace's equation applies in charge-free regions; the uniqueness theorem guarantees that a potential satisfying the same boundary conditions is the physical solution; and the method of images uses fictitious charges to construct that solution conveniently.