

Curl of an Electrostatic Field, Conservative Nature, Electric Potential

What is Curl?

The curl of a vector field measures the tendency of the field to produce circulation around a point.

For a vector field E , $\nabla \times E$ is called the **curl of the electric field**.

$$\text{In Cartesian coordinates, } \nabla \times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

Curl of an Electrostatic Field

For a point charge q , $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$ Since $\hat{r} = \frac{\vec{r}}{r}$ we get $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$

$$\text{Now, } \nabla \times \vec{E} = \frac{q}{4\pi\epsilon_0} \nabla \times \left(\frac{\vec{r}}{r^3} \right)$$

Write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = (x^2 + y^2 + z^2)^{1/2}$

$$\text{Therefore, } \frac{\vec{r}}{r^3} = \frac{x}{r^3}\hat{i} + \frac{y}{r^3}\hat{j} + \frac{z}{r^3}\hat{k} \text{ Hence, } \nabla \times \vec{E} = \frac{q}{4\pi\epsilon_0} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix}$$

Consider the x -component: $\frac{\partial}{\partial y} \left(\frac{z}{r^3} \right) - \frac{\partial}{\partial z} \left(\frac{y}{r^3} \right)$

Since $\frac{\partial}{\partial y} \left(\frac{z}{r^3} \right) = -\frac{3yz}{r^5}$ and $\frac{\partial}{\partial z} \left(\frac{y}{r^3} \right) = -\frac{3yz}{r^5}$

$$\text{Therefore, } \frac{\partial}{\partial y} \left(\frac{z}{r^3} \right) - \frac{\partial}{\partial z} \left(\frac{y}{r^3} \right) = 0$$

Similarly, y & z -components also 0.

Therefore, $\nabla \times E = 0$ for an electrostatic field.

This tells us that an electrostatic field is a **conservative field**.

This property allows us to define a scalar quantity called **electric potential**.

Integral Form of the Result

Using Stokes' theorem, $\oint_C E \cdot dl = \int_S (\nabla \times E) \cdot dS$.

$$\text{Since } \nabla \times E = 0, \text{ we get } \oint_C E \cdot dl = 0$$

Therefore, the work done by an electrostatic field around a closed path is zero.

Conservative Nature of Electrostatic Field

A vector field is called **conservative** if its line integral around any closed path is zero.

Since $\oint E \cdot dl = 0$ the electrostatic field is a **conservative field**.

What does this mean physically?

Suppose a charge is moved from point A to point B . The work done by the electrostatic field does **not depend on the path followed**. It depends only on the initial and final positions.

Therefore, $W_{A \rightarrow B}$ is path independent

Work Done in Moving a Charge

Suppose a test charge q is moved from A to B .

$$\text{The work done by the electric field is } W_{A \rightarrow B} = q \int_A^B E \cdot dl$$

Because the field is conservative, the result is independent of the path.

$$\text{For a closed path, } W_{\text{closed}} = 0$$

This is analogous to the gravitational field.

Definition of Electric Potential

The **electric potential at a point** is defined as the work done by an external agent in bringing a unit positive test charge from infinity to that point, without acceleration.

Mathematically,
$$V(P) = \frac{W_{\text{external}}}{q}$$

If the potential at infinity is chosen to be zero, $V(\infty) = 0$

then the potential at P is
$$V(P) = - \int_{\infty}^P E \cdot dl$$

The SI unit of electric potential is **volt (V)**. $1 \text{ volt} = 1 \text{ joule/coulomb}$

Potential Difference Between two points A and B is
$$V_B - V_A = - \int_A^B E \cdot dl$$

Therefore, the work done in moving a charge q from A to B is
$$W_{A \rightarrow B} = q(V_A - V_B)$$

Relation Between Electric Field and Potential

From $V_B - V_A = - \int_A^B E \cdot dl$, we obtain the differential relation
$$dV = -E \cdot dl$$

Therefore,
$$E = -\nabla V$$

In Cartesian coordinates,
$$E = - \left(\frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k} \right)$$

For spherical symmetry, $V = V(r)$, so
$$E_r = - \frac{dV}{dr}$$

Thus, the electric field is obtained by taking the **negative gradient of potential**.

Potential Due to a Uniformly Charged Spherical Shell

Consider a thin spherical shell of radius R , carrying a total charge Q , uniformly distributed over its surface. We want to find the electric potential:

1. outside the shell,
2. on the surface,
3. inside the shell.

Electric Field of a Spherical Shell

$$E = \begin{cases} 0, & r < R \\ \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}, & r > R \end{cases}$$

Potential Outside the Shell $V(\infty) = 0$.

For $r > R$, $V(r) = -\int_{\infty}^r E \cdot dl$.

Therefore, $V(r) = -\int_{\infty}^r \frac{1}{4\pi\epsilon_0} \frac{Q}{r'^2} dr'$.

Hence, $V(r) = \frac{Q}{4\pi\epsilon_0 r}$.

Therefore, $V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}, r \geq R$

So outside the spherical shell, the potential is exactly the same as that of a point charge Q at the centre.

Potential on the Surface: $r = R$,

we obtain $V(R) = \frac{Q}{4\pi\epsilon_0 R}$

Potential Inside the Shell $r < R$, the electric field is zero.

Therefore, $\frac{dV}{dr} = -E = 0$. Hence, $V = \text{constant}$.

Since potential is continuous at the surface, $V(r) = V(R)$.

Therefore, $V(r) = \frac{Q}{4\pi\epsilon_0 R}, r < R$

Final result for a spherical shell

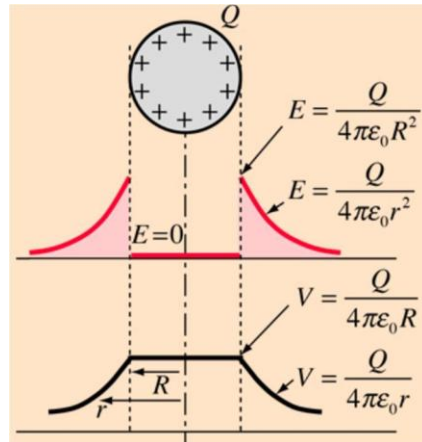
$$V(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r}, & r \geq R \\ \frac{Q}{4\pi\epsilon_0 R}, & r < R \end{cases}$$

Potential Due to a Uniformly Charged Solid Sphere

Consider a solid non-conducting sphere of radius R with total charge Q , uniformly distributed throughout its volume. The volume charge density is $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$ or $\rho = \frac{3Q}{4\pi R^3}$

We want to calculate the potential both outside and inside the sphere.

Electric Field Outside the Solid Sphere: $E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$



Potential Outside the Solid Sphere: we have $V(r) = - \int_{\infty}^r E \, dr$.

Therefore, $V(r) = \frac{Q}{4\pi\epsilon_0 r}$, $r \geq R$ At the surface, $V(R) = \frac{Q}{4\pi\epsilon_0 R}$

Electric Field Inside the Solid Sphere: $E(r) = \frac{Q}{4\pi\epsilon_0 R^3} r$

Potential Inside the Solid Sphere

Now, $dV = -E \, dr$. We know the potential at the surface: $V(R) = \frac{Q}{4\pi\epsilon_0 R}$.

For an interior point $r < R$, $V(r) - V(R) = - \int_R^r E(r') \, dr'$.

Substituting $E(r') = \frac{Q}{4\pi\epsilon_0 R^3} r'$,

we get, $V(r) - V(R) = - \frac{Q}{4\pi\epsilon_0 R^3} \int_R^r r' \, dr'$.

Therefore, $V(r) - V(R) = - \frac{Q}{4\pi\epsilon_0 R^3} \left[\frac{r'^2}{2} \right]_R^r$.

Thus, $V(r) - V(R) = \frac{Q}{8\pi\epsilon_0 R^3} (R^2 - r^2)$.

Using $V(R) = \frac{Q}{4\pi\epsilon_0 R}$,

we obtain $V(r) = \frac{Q}{4\pi\epsilon_0 R} + \frac{Q}{8\pi\epsilon_0 R^3} (R^2 - r^2)$.

Simplifying, $V(r) = \frac{Q}{8\pi\epsilon_0 R^3} (3R^2 - r^2)$ for $r < R$.

For Uniformly Charged Solid Sphere

$$V(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r}, & r \geq R \\ \frac{Q}{8\pi\epsilon_0 R^3} (3R^2 - r^2), & r < R \end{cases}$$

At the centre, $r = 0$.

Therefore, $V(0) = \frac{3Q}{8\pi\epsilon_0 R}$ At the surface, $V(R) = \frac{Q}{4\pi\epsilon_0 R}$

Comparison: Spherical Shell and Solid Sphere

Quantity	Spherical shell Uniformly charged solid sphere	
Outside electric field	$\frac{Q}{4\pi\epsilon_0 r^2}$	$\frac{Q}{4\pi\epsilon_0 r^2}$
Inside electric field	0	$\frac{Qr}{4\pi\epsilon_0 R^3}$
Surface potential	$\frac{Q}{4\pi\epsilon_0 R}$	$\frac{Q}{4\pi\epsilon_0 R}$
Inside potential	Constant	Varies with r
Centre potential	$\frac{Q}{4\pi\epsilon_0 R}$	$\frac{3Q}{8\pi\epsilon_0 R}$

Calculation of Electric Field from Potential

The electric field is obtained from the potential using $\vec{E} = -\nabla V$

Cartesian Coordinates: If $V = V(x, y, z)$,

then
$$E = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

Spherically Symmetric Potential: If the potential depends only on r , $V = V(r)$,

then
$$E_r = -\frac{dV}{dr} \quad \text{and} \quad E = -\frac{dV}{dr}\hat{r}$$

This is particularly useful for spherical charge distributions.

Example: Find E from Potential of a Charged Sphere

For the outside region, $V(r) = \frac{Q}{4\pi\epsilon_0 r}$. We know, $E_r = -\frac{dV}{dr}$.

Now, $\frac{dV}{dr} = -\frac{Q}{4\pi\epsilon_0 r^2}$. Hence,
$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

Which agrees with the result obtained from Gauss's law.

Inside a Uniformly Charged Solid Sphere: Inside, $V(r) = \frac{Q}{8\pi\epsilon_0 R^3}(3R^2 - r^2)$.

$$\frac{dV}{dr} = -\frac{Qr}{4\pi\epsilon_0 R^3}. \quad \text{Hence,} \quad E_r = \frac{Qr}{4\pi\epsilon_0 R^3}$$

Again, we obtain the same result as from Gauss's law.

Uniform spherical shell
$$V = \begin{cases} \frac{Q}{4\pi\epsilon_0 r}, & r \geq R \\ \frac{Q}{4\pi\epsilon_0 R}, & r < R \end{cases}$$

Uniform solid sphere
$$V = \begin{cases} \frac{Q}{4\pi\epsilon_0 r}, & r \geq R \\ \frac{Q}{8\pi\epsilon_0 R^3}(3R^2 - r^2), & r < R \end{cases}$$

Short Examination Questions

1. Show that the curl of an electrostatic field is zero.
2. Explain why an electrostatic field is conservative.
3. Define electric potential.
4. Derive the relation $\vec{E} = -\nabla V$.
5. Derive the potential due to a uniformly charged spherical shell.
6. Show that the potential inside a uniformly charged spherical shell is constant.
7. Derive the potential inside a uniformly charged solid sphere.
8. Find the potential at the centre of a uniformly charged solid sphere.
9. Explain how electric field can be calculated from electric potential.

Why can the electric potential be non-zero when the electric field is zero?

Because $E = -\frac{dV}{dr}$. If $E = 0$, then $\frac{dV}{dr} = 0$, which means that V is **constant**, not necessarily zero.

For a spherical shell inside the shell
$$E = 0, V = \frac{Q}{4\pi\epsilon_0 R}$$